

UNIFORM (M)-CONDITION AND STRONG MILNOR FIBRATIONS

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ABSTRACT. In this paper we study the Milnor fibrations associated to real analytic map germs $\psi : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^2, 0)$ with isolated critical point at $0 \in \mathbb{R}^m$. The main result relates the existence of called Strong Milnor fibrations with a transversality condition of a convenient family of analytic varieties with isolated critical points at the origin $0 \in \mathbb{R}^m$, obtained by projecting the map germ ψ in the family $L_{-\theta}$ of all lines through the origin in the plane \mathbb{R}^2 .

1. INTRODUCTION

In [Mi] Milnor proved that if

$$\psi : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0),$$

is the germ of a holomorphic function with a critical point at 0, then for every sufficiently small $\epsilon > 0$ the map $\frac{\psi}{\|\psi\|} : S_\epsilon^{2n+1} \setminus K_\epsilon \rightarrow S^1$ is the projection map of a smooth locally trivial fibre bundle, where $K_\epsilon = \psi^{-1}(0) \cap S_\epsilon^{2n+1}$ is the link of singularity at 0. This is the Milnor fibration for holomorphic singularities functions germs.

Milnor also proved in the last chapter of his book a fibration theorem for real singularities. He showed that if

$$\psi : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^p, 0), m \geq p \geq 2,$$

is a real analytic map germ whose derivative $D\psi$ has rank p on a punctured neighborhood of $0 \in \mathbb{R}^m$, then there exists $\epsilon > 0$ and $\eta > 0$ sufficiently small with $0 < \eta \ll \epsilon < 1$, such that considering $E := B_\epsilon^m(0) \cap \psi^{-1}(S_\eta^{p-1})$, $B_\epsilon^m(0)$ the open ball centered in $0 \in \mathbb{R}^m$ and radius ϵ , we have that $\psi|_E : E \rightarrow S_\eta^{p-1}$ is a smooth locally trivial fibre bundle.

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Milnor also proved the existence of a diffeomorphism that pushes E to $S_\epsilon^{m-1} \setminus N_{K_\epsilon}$, where N_{K_ϵ} denotes a tubular neighborhood of the link K_ϵ in S_ϵ^{m-1} . Moreover, this fibre bundle can be extended to the complement of the link in the sphere $S_\epsilon^{m-1} \setminus K_\epsilon$, with each fiber being the interior of a compact manifold bounded by K_ϵ . But, in all these constructions we cannot guarantee that the map $\frac{\psi}{\|\psi\|}$ is the projection of the fibration, as it is easily shown by the example below.

Example 1.1. [Mi, page 99]

$$\begin{cases} P = x \\ Q = x^2 + y(x^2 + y^2) \end{cases}$$

Definition 1.2. [RSV]

Let $\psi : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^p, 0)$, $m \geq p \geq 2$, be a map germ with isolated singularity at the origin. If for all $\epsilon > 0$ sufficiently small, the map $\phi = \frac{\psi}{\|\psi\|} : S_\epsilon^{m-1} \setminus K_\epsilon \rightarrow S^{p-1}$, is a projection of a smooth locally trivial fibre bundle, where K_ϵ is the link of singularity at 0, we say that the map germ satisfies the **Strong Milnor condition** at $0 \in \mathbb{R}^m$.

The problem of studying real isolated singularities for which the map $\frac{\psi}{\|\psi\|}$ extends as a smooth projection of the fibre bundle $S_\epsilon^{m-1} \setminus K_\epsilon \rightarrow S^{p-1}$, as in the holomorphic case, was first studied by A. Jacquemard in [Ja], [Ja1], by J. Seade, Ruas and Verjovsky in [RSV], and by the author and Ruas in [RS].

The Jacquemard's approach [Ja] was the following: considering $\psi = (P, Q) : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^2, 0)$ an analytic real map germ with isolated singularity at $0 \in \mathbb{R}^m$, he gave two conditions which were sufficient to guarantee that the map $\frac{\psi}{\|\psi\|}$ extends to all $S_\epsilon^{m-1} \setminus K_\epsilon$ as a smooth projection map of a locally trivial fiber bundle over S^1 , i.e, the function germ ψ satisfy the Strong Milnor condition at origin. The first condition (A) is geometric: the angle between the gradient vector fields ∇P and ∇Q has an upper bound smaller than 1; the second condition (B) is algebraic: the Jacobian ideals of P and Q have the same integral closure in the local ring of real analytic function germs at $0 \in \mathbb{R}^m$. With these tools the author recovered some main ideas given by Milnor on his book [Mi] to construct the locally trivial fibre bundle.

In another direction, using stratification theory and singularity theory, in [RS] the authors proved that the Jacquemard's conditions are

not necessary for the existence of a Milnor fibration. The result were the following:

still considering $\psi = (P, Q) : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^2, 0)$ a real analytic map germ, with isolated singularity at $0 \in \mathbb{R}^m$, and $\Psi(x, t)$ a convenient family of functions associated to map ψ (called Seade's family, see Definition 2.4, page 5), $X = \Psi^{-1}(0) \setminus \{0\} \times \mathbb{R}$ and $Y = \{0\} \times \mathbb{R}$ a stratification of analytic variety $\Psi^{-1}(0)$, holds:

Theorem 1.3. [RS] *If the real analytic map germ $\psi = (P, Q) : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^2, 0)$ satisfies the Jacquemard hypotheses, then the pair (X, Y) as above satisfies the Verdier's (w) -condition.*

It is well known that, in real and complex subanalytic settings we have these sequences of implications: Verdier's (w) -condition \Rightarrow Kuo's ratio test \Rightarrow (b)-Whitney condition \Rightarrow Bekka's (c)-condition.

Theorem 1.4. [RS] *If the pair (X, Y) , as above, satisfy the Bekka's (c)-condition on $\mathbb{R}^m \times \mathbb{R}$ with respect the control function $\rho(x, \theta) = \sum_{i=1}^n x_i^2$ (or, in another words, the pair (X, Y) satisfies the A_ρ -Thom condition), then the map $\frac{\psi}{\|\psi\|}$ extends as a smooth projection of locally trivial fibre bundle $S_\epsilon^{m-1} \setminus K_\epsilon \rightarrow S^1$.*

In the example below, is easy to see that, the pair (X, Y) associated to Seade's family satisfies the A_ρ -Thom condition for $\rho(x, y, \theta) = x^2 + y^2$ and $\rho|_X$ is a submersion, i.e. the pair (X, Y) satisfies (c)-regularity condition (see definition 2.2), but does not satisfy the Jacquemard hypothesis (B), in this case), showing that the Jacquemard hypotheses is stronger than the hypothesis given by the authors in Theorem 1.4.

Example 1.5.

$$\begin{cases} P = xy \\ Q = x^2 - y^4 \end{cases}$$

In this work, still using the family $\Psi(x, \theta)$, we give another point of view to get the Strong Milnor fibration. Actually, following the approach given by Milnor in [Mi] and by A. Jacquemard in [Ja], we describe a condition weaker than (c)-regularity, as given in [RS]. The main Theorem is:

Theorem 1.6. *Let $\psi = (P, Q) : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^2, 0)$ be a real analytic map germ, with isolated critical point at $0 \in \mathbb{R}^m$, and $\Psi(x, \theta)$ the associated Seade's family for the map ψ . Suppose that for all $x \in U \setminus \{0\}$, where U is an open domain of ψ , we have:*

$|\langle \frac{\nabla_x \Psi_\theta(x)}{\|\nabla_x \Psi_\theta(x)\|}, \frac{x}{\|x\|} \rangle| \leq 1 - \rho; 0 < \rho \leq 1; \forall \theta \in \mathbb{R}$. Then, ψ satisfies the Strong Milnor condition at $0 \in \mathbb{R}^m$, i.e. there exist $\epsilon_0 > 0$, sufficiently small, such that $\forall \epsilon, 0 < \epsilon \leq \epsilon_0$, the projection map $\frac{\psi}{\|\psi\|} : S_\epsilon^{m-1} \setminus K_\epsilon \rightarrow S^1$ is a smooth locally trivial fibre bundle.

2. DEFINITION AND BASIC RESULTS

We briefly recall some definitions and basic results. For more details see [BK], [B1].

Let M be a smooth Riemannian manifold, X and Y submanifolds of M , such that $Y \subset \overline{X}$. Let (T_Y, π, ρ) be a tubular neighbourhood of Y in M together with a projection $\pi : T_Y \rightarrow Y$, associated to a smooth non-negative control function ρ with $\rho^{-1}(0) = Y$ and $\nabla \rho(x) \in \ker(d\pi(x))$.

Definition 2.1. [BK]:

The pair (X, Y) satisfies condition (m) if there exists a real number $\epsilon > 0$ such that

$$\begin{aligned} (\pi, \rho) |_{X \cap T_Y^\epsilon} : X \cap T_Y^\epsilon &\rightarrow Y \times \mathbb{R} \\ x &\mapsto (\pi(x), \rho(x)) \end{aligned}$$

is a submersion, where $T_Y^\epsilon := \{x \in T_Y / \rho(x) < \epsilon\}$.

Geometrically, the condition (m) says that the submanifold X is transverse to level $\rho = c$ inside the open tubular neighborhood T_Y^ϵ .

Definition 2.2. (Bekka's condition) [B2]

We say that a pair of strata (X, Y) satisfies (c)–regularity condition with respect a smooth non-negative control function $\rho : M \rightarrow \mathbb{R}^+$, if the following holds:

- i) $\rho^{-1}(0) = Y$;
- ii) $\rho |_X$ is a submersion;
- iii) Let $\{x_i\}$ a sequence of points in X such that $\{x_i\} \rightarrow y \in Y$ and $\ker(d_{x_i} \rho |_X) \rightarrow \tau$; then $T_y Y \subset \tau$.

Considering $Star(Y) = \{X : X \text{ is statum such that } \overline{Y} \subset X\}$, in a general way, the property ii) of definition says that $\rho |_{Star(Y)}$ is a stratified map and iii) says that $\rho |_{Star(Y)}$ is a Thom map. The following proposition is indeed a geometric easy way to see the item iii).

Proposition 2.3. [B2] *The property iii) of the above definition is equivalent to*

$$\lim_{x \rightarrow y} \Pi_Y \left(\frac{\text{grad}_x(\rho|_X)}{\|\text{grad}_x(\rho|_X)\|} \right) = 0$$

where Π_Y is the orthogonal projection on $T_y Y$ and $\mathbf{0} \in T_y Y$.

Now let $F : \mathbb{R}^m \times \mathbb{R}, 0 \times \mathbb{R} \rightarrow \mathbb{R}, 0$ be a one-parameter family of function-germs, $F(x, \theta) = F_\theta(x)$, $X := F^{-1}(0) \setminus (0 \times \mathbb{R}) \subset \mathbb{R}^m \times \mathbb{R}$, $Y := 0 \times \mathbb{R}$ and $X_\theta = F_\theta^{-1}(0) \subset \mathbb{R}^m, 0$.

We say that the family $F(x, \theta)$ has ρ -Milnor's radius uniformly, if there exist $\epsilon_0 > 0$ such that the pair (X, Y) , defined as above, satisfies condition (m), with respect some control function ρ .

Remark.

1. If the definition holds for some $\epsilon_0 > 0$, it also holds for all ϵ , $0 < \epsilon \leq \epsilon_0$;
2. Using the control function $\rho(x_1, \dots, x_m, \theta) = \sum_{i=1}^m x_i^2$ in \mathbb{R}^{m+1} , this definition says that the strata X is transverse to all Euclidean cylinder into the tubular neighborhood $T_Y^{\epsilon_0}$. More precisely, the manifolds $X_\theta = F_\theta^{-1}(0)$ are transverse to spheres S_ϵ^{m-1} , for all $0 < \epsilon \leq \epsilon_0$.

Finally we recall Seade's method given in [S], [RS]. Consider a real analytic map germ $\psi : \mathbb{R}^m, 0 \rightarrow \mathbb{R}^2, 0$ and identify \mathbb{R}^2 with \mathbb{C} , we have $\psi(x) = (P(x), Q(x)) \approx P(x) + iQ(x)$, where $i^2 = -1$. Let $\pi_\theta : \mathbb{C} \rightarrow L_\theta$ be the orthogonal projection to the line L_θ through the origin, forming angle θ with the horizontal axis in \mathbb{C} and take the composition $\Psi(x, \theta) = \pi_\theta \circ \psi(x)$.

Lemma 2.4. [S] *Let $U \subseteq \mathbb{R}^m$ be a neighborhood of 0 such that for every $x \in U \setminus \{0\}$, ψ has maximal rank at x . Then the following hold:*

- (i) $U = \cup_\theta (M_\theta \cap U)$, $0 \leq \theta < \pi$.
- (ii) $M = \cap_\theta M_\theta = M_{\theta_1} \cap M_{\theta_2}$, where $M = \psi^{-1}(0)$, $\theta_1 \neq \theta_2$, $\theta_1, \theta_2 \in [0, \pi)$.
- (iii) For each $\theta \in [0, \pi)$, $M_\theta = E_\theta \cup M \cup E_{\theta+\pi}$, where $E_\alpha = \tilde{\phi}^{-1}(e^{i\alpha})$ and $M = \psi^{-1}(0)$, with $\tilde{\phi} : U \setminus M \rightarrow S^1$, $\tilde{\phi}(x) = i \frac{\overline{\psi(x)}}{\|\psi(x)\|}$.
- (iv) For each $\theta \in [0, \pi)$, $M_\theta^* = M_\theta \setminus \{0\}$ is a real smooth submanifold of real codimension 1 of $U \setminus \{0\}$, given by the union of E_θ , $E_{\theta+\frac{\pi}{2}}$ and $M \setminus \{0\}$.

Definition 2.5. The family $\Psi : (\mathbb{R}^m \times \mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$, $\Psi(x, \theta) = \pi_\theta \circ \psi(x)$, defined as above, will be called the Seade's family associated to the map germ $\psi = (P, Q)$.

3. TOOLS

In what follows let $\psi = (P, Q) : \mathbb{R}^m, 0 \rightarrow \mathbb{R}^2, 0$ be a real analytic map germ with isolated critical point at $0 \in \mathbb{R}^m$, $U \ni 0$ some open domain of ψ with the decomposition given by Lemma 2.4 and $\Psi : \mathbb{R}^m \times \mathbb{R}, 0 \rightarrow \mathbb{R}, 0$ the associated Seade's family to the real analytic map germ ψ , satisfying the hypothesis of Theorem 1.6.

In this section we prove two preliminary lemmas. In the first we show that the projection $\frac{\psi}{\|\psi\|}$ does not have critical point in $S_\epsilon^{m-1} \setminus K_\epsilon$ for all ϵ sufficiently small. Using the hypothesis of Theorem 1.6, it will be possible to guarantee the existence of $\epsilon_0 > 0$ sufficiently small, such that the manifolds $M_\theta \setminus \{0\}$ are transverse to S_ϵ^{m-1} for all $\epsilon, 0 < \epsilon \leq \epsilon_0$, and for all θ .

Lemma 3.1. *There exists $\epsilon_0 > 0$, sufficiently small, such that for all $x \in B_{\epsilon_0}^m(0) \setminus \{\psi^{-1}(0)\} \subset U$, the vector fields $\nabla_x \Psi_\theta(x)$ and $\gamma(x) := P(x)\nabla Q(x) - Q(x)\nabla P(x)$ are parallel.*

Proof. Let $B_{\epsilon_0}^m(0) = \cup[M_\theta \cap B_{\epsilon_0}^m(0)]$. For each $x \in B_{\epsilon_0}^m(0) \setminus \{\psi^{-1}(0)\}$, $\exists \theta \in \mathbb{R}$ such that $\cos(\theta)P(x) = \sin(\theta)Q(x)$, since x belongs to some M_θ .

So, consider the following cases:

- 1) $\sin(\theta)Q(x) \neq 0$;
- 2) $\sin(\theta)Q(x) = 0$.

If 1) holds, we have

$$P(x) = \frac{\sin(\theta)}{\cos(\theta)}Q(x) \text{ and } \gamma(x) = \frac{\sin(\theta)}{\cos(\theta)}Q(x)\nabla Q(x) - Q(x)\nabla P(x) = -\frac{Q(x)}{\cos(\theta)}(\cos(\theta)\nabla P(x) - \sin(\theta)\nabla Q(x)). \text{ Then, } \gamma(x) = -\frac{Q(x)}{\cos(\theta)}\nabla \Psi_\theta(x).$$

If we have the particular case 2) $\sin(\theta)Q(x) = 0$, consider again two situations:

- i) If $\sin(\theta) = 0$ then $\cos(\theta) \neq 0$ and $P(x) = 0$. Since $x \in B_{\epsilon_0}^m(0) \setminus \psi^{-1}(0)$, then $\gamma(x) = -Q(x)\nabla P(x)$ and $\nabla_x \Psi_\theta(x) = \cos(\theta)\nabla P(x)$.
- ii) If $Q(x) = 0$ then $\cos(\theta) = 0$ and $P(x) \neq 0$, since $x \in B_{\epsilon_0}^m(0) \setminus \psi^{-1}(0)$. Then, $\sin(\theta) \neq 0$, $\gamma(x) = P(x)\nabla Q(x)$ and $\nabla_x \Psi_\theta(x) = -\sin(\theta)\nabla Q(x)$.

Therefore, in both cases we have $\gamma(x)$ is parallel to $\nabla_x \Psi_\theta(x)$. □

In [Ja] the author proved that the critical points of the map $\frac{\psi}{\|\psi\|} : S_\epsilon^{m-1} \setminus K_\epsilon \rightarrow S^1$ are precisely the points $x \in S_\epsilon^{m-1}$ such that the vector fields $\gamma(x) = P(x)\nabla Q(x) - Q(x)\nabla P(x)$ and x are parallel. However, the hypothesis of the main theorem implies that the vector fields x and $\nabla_x \Psi_\theta(x)$ are transversal. So, using Lemma 3.1 above we have that the projection $\frac{\psi}{\|\psi\|} : S_\epsilon^{m-1} \setminus K_\epsilon \rightarrow S^1$ is a submersion, for all $\epsilon > 0$ sufficiently small.

In the following result we construct a smooth vector fields in $S_\epsilon^{m-1} \setminus K_\epsilon$, whose solution do not fall in the empty K_ϵ , for finite time. This result will guarantee that the projection $\frac{\psi}{\|\psi\|}$ is a onto submersion.

Lemma 3.2. *There exists a smooth vector fields ω tangent to $S_\epsilon^{m-1} \setminus K_\epsilon$, such that:*

- (i) $\langle \omega(x), \frac{P(x)\nabla Q(x) - Q(x)\nabla P(x)}{P^2(x) + Q^2(x)} \rangle = 1$,
- (ii) $|\langle \omega(x), \frac{P(x)\nabla P(x) + Q(x)\nabla Q(x)}{P^2(x) + Q^2(x)} \rangle| \leq M$; $M > 0$, for each $\epsilon > 0$, sufficiently small.

Proof. For each $x \in S_\epsilon^{m-1} \setminus K_\epsilon$ define $u(x) := \gamma(x) - \langle \gamma(x), \frac{x}{\|x\|} \rangle \cdot \frac{x}{\|x\|}$, the projection of $\gamma(x)$ in $T_x(S_\epsilon^{m-1} \setminus K_\epsilon)$. Under the hypotheses of the main Theorem and Lemma 3.1 above, this vector field is smooth and never zero.

Let $w(x) := (\frac{P^2(x) + Q^2(x)}{\|u(x)\|}) \cdot \frac{u(x)}{\|u(x)\|}$. So we have:

$$\begin{aligned} \text{i) } \langle w(x), \frac{P(x)\nabla Q(x) - Q(x)\nabla P(x)}{P(x)^2 + Q(x)^2} \rangle &= \langle \frac{u(x)}{\|u(x)\|^2}, \gamma(x) \rangle = \\ \frac{1}{\|u(x)\|^2} \langle u(x), \gamma(x) - \langle \gamma(x), \frac{x}{\|x\|} \rangle \cdot \frac{x}{\|x\|} \rangle &= \frac{1}{\|u(x)\|^2} \langle u(x), u(x) \rangle = 1, \end{aligned}$$

where the second equality we use the fact $u(x) \perp \langle \gamma(x), \frac{x}{\|x\|} \rangle \cdot \frac{x}{\|x\|}$. This proves the first statement.

$$\begin{aligned} \text{ii) } \langle w(x), \frac{P(x)\nabla P(x) + Q(x)\nabla Q(x)}{P(x)^2 + Q(x)^2} \rangle &= \frac{1}{2} \langle w(x), \frac{\nabla(\|\psi(x)\|^2)}{P(x)^2 + Q(x)^2} \rangle = \\ \frac{1}{2} \langle \frac{u(x)}{\|u(x)\|^2}, \nabla(\|\psi(x)\|^2) \rangle &\leq \frac{1}{2} \frac{\|u(x)\| \cdot \|\nabla(\|\psi(x)\|^2)\|}{\|u(x)\|^2} = \frac{1}{2} \frac{\|\nabla(\|\psi(x)\|^2)\|}{\|u(x)\|}. \end{aligned}$$

Since $\|u(x)\|^2 = \|\gamma(x)\|^2 - \langle \gamma(x), \frac{x}{\|x\|} \rangle^2 = \|\gamma(x)\|^2(1 - \langle \frac{\gamma(x)}{\|\gamma(x)\|}, \frac{x}{\|x\|} \rangle^2) >$

$$> \rho^2 \|\gamma(x)\|^2 \text{ then,} \\ |\langle w(x), \frac{\nabla(\|\psi(x)\|^2)}{P(x)^2 + Q(x)^2} \rangle| \leq \frac{1}{2} \frac{\|\nabla(\|\psi(x)\|^2)\|}{\|u(x)\|} < \frac{1}{2\rho} \frac{\|\nabla(\|\psi(x)\|^2)\|}{\|\gamma(x)\|}.$$

It is enough to verify that for all $x \in S_\epsilon^{m-1} \setminus K_\epsilon$, it is possible to get upper bounds for the last expression by using the curve selection lemma.

For this, consider $\delta > 0$ a real number sufficiently small, and a non-constant real analytic curve $\alpha : [0, \delta) \rightarrow S_\epsilon^{m-1}$, with $\alpha(t) \in S_\epsilon^{m-1} \setminus K_\epsilon$ and $\alpha(0) \in K_\epsilon$. Using the Taylor expansion we have:

$$\begin{cases} \alpha(s) = \alpha_0 + \alpha_1 s^n + \dots; n \geq 1, \alpha_0 \in K_\epsilon. \\ P(\alpha(s)) = P_0 + P_1 s^r + \dots; r \geq 1; \\ \nabla P(\alpha(s)) = a_0 + a_1 s^l + \dots, l \geq 1, a_0 \neq 0; \\ Q(\alpha(s)) = Q_0 + Q_1 s^k + \dots, k \geq 1; \\ \nabla Q(\alpha(s)) = b_0 + b_1 s^p + \dots, p \geq 1, b_0 \neq 0. \end{cases}$$

Since $\alpha_0 \in K_\epsilon = S_\epsilon^{m-1} \cap \psi^{-1}(0) \implies P(\alpha(0)) = P_0 = 0$ and $Q(\alpha(0)) = Q_0 = 0$. Therefore

$$\begin{cases} P(\alpha(s)) = P_1 s^r + \dots, r \geq 1, P_1 \neq 0 \\ Q(\alpha(s)) = Q_1 s^k + \dots, k \geq 1, Q_1 \neq 0 \end{cases}$$

Since $\Sigma(P, Q) = \{0\}$ the vectors $\nabla P(\alpha(0))$ and $\nabla Q(\alpha(0))$ are not parallel outside of origin.

Then, $\frac{\|\nabla(\|\psi(\alpha(s))\|^2)\|^2}{\|\gamma(\alpha(s))\|^2} = \frac{\|(P_1 s^r + \dots)(a_0 + a_1 s^l + \dots) + (Q_1 s^k + \dots)(b_0 + b_1 s^p + \dots)\|^2}{\|(P_1 s^r + \dots)(b_0 + b_1 s^p + \dots) - (Q_1 s^k + \dots)(a_0 + a_1 s^l + \dots)\|^2} = \frac{\|P_1 a_0 s^r + Q_1 b_0 s^p + \dots\|^2}{\|P_1 b_0 s^r - Q_1 a_0 s^p + \dots\|^2} = \frac{\|P_1 a_0 s^r + Q_1 b_0 s^p\|^2 + s^2(U(s))}{\|P_1 b_0 s^r - Q_1 a_0 s^p\|^2 + s^2(V(s))}$; for some analytic functions $U(s)$, $V(s)$ in variable s .

The last expression has a upper bound for s small enough, if we take any natural r, p with $r \neq p$. Now it remains to check if $r = p$ and $P_1 b_0 - Q_1 a_0 = 0$ because, in this particular case, the order of denominator can be bigger than the order of numerator when s goes to zero. It means that the last expression above goes to infinity.

$$\text{But } P_1 b_0 - Q_1 a_0 = 0 \iff b_0 = \frac{Q_1}{P_1} a_0 \text{ and}$$

$$\begin{cases} \nabla P(\alpha(s)) = a_0 + a_1 s^l + \dots \\ \nabla Q(\alpha(s)) = \frac{Q_1}{P_1} a_0 + b_1 s^k + \dots \end{cases}$$

Then, $\nabla P(\alpha(0))/\nabla Q(\alpha(0))$ in S_ϵ^{m-1} , contradicting $\sum(P, Q) = \{0\}$. \square

4. MAIN RESULT

Proposition 4.1. *Let $\psi : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^2, 0)$ with isolated critical point at $0 \in \mathbb{R}^m$, such that the associated Seade's family satisfies the hypothesis of main theorem, then there exist ϵ_0 such that, for all $0 < \epsilon \leq \epsilon_0$, the map $\frac{\psi}{\|\psi\|} : S_\epsilon^{m-1} \setminus K_\epsilon \rightarrow S^1$ is a smooth projection of a locally trivial fibre bundle.*

Proof. Now we are ready to construct the prove of Theorem 1.6.

Under it hypothesis and by Lemma 3.2, there exists a smooth vector fields ω tangent to $S_\epsilon^{m-1} \setminus K_\epsilon$, satisfying conditions i) and ii) of lemma. Now taking the flow $\phi_t(x)$ of $\omega \in T_x(S_\epsilon^{m-1} \setminus K_\epsilon)$, with $\phi_0(x) = x$, and considering $\frac{\psi(x)}{\|\psi(x)\|} = e^{i\theta(x)}$, we have $\theta(x) = \text{Re}(-i \ln \psi(x))$ and

$$\frac{d}{dt}(\theta(\phi_t(x))) = \langle \dot{\phi}_t(x), \frac{P(\phi_t(x))\nabla Q(\phi_t(x)) - Q(\phi_t(x))\nabla P(\phi_t(x))}{P^2(\phi_t(x)) + Q^2(\phi_t(x))} \rangle = 1$$

then, $\theta(\phi_t(x)) = t + c$; $\theta(\phi_t(x)) = t + \theta_0$, where $\theta(\phi_0(x)) = \theta(x) = \theta_0$. it the initial value.

Therefore, if the flow ϕ_t is well defined $\forall t \in \mathbb{R}$, we have $\frac{\psi}{\|\psi\|} : S_\epsilon^{m-1} \setminus K_\epsilon \rightarrow S^1$ is an onto smooth submersion over S^1 , because the projection function $\frac{\psi}{\|\psi\|}$ wrap $\phi_t(x)$ around S^1 , all time t .

If for some fixed $t_0 \in \mathbb{R}$, $t \rightarrow t_0 \implies \phi_t \rightarrow K_\epsilon$, then $\|\psi(\phi_t)\| \rightarrow 0$ and $\log \|\psi(\phi_t)\| \rightarrow \infty$.

But, $|\frac{d}{dt}(\log(\|\psi(\phi_t)\|^2))| = |\langle \dot{\phi}_t, \frac{\nabla(\|\psi(\phi_t)\|^2)}{P(\phi_t)^2 + Q(\phi_t)^2} \rangle| \leq M$ has bounded derivative, so this flow is well defined for all $t \in \mathbb{R}$.

Now using the same idea of [Mi, page 43], see also [Ja1, page 22], consider $\pi = \frac{\psi}{\|\psi\|}$, $x_0 \in S_\epsilon^{m-1} \setminus K_\epsilon$ fixed, define $h_{x_0}(t) = \phi_{x_0}(t)$. For each t ,

it is well known that $h_t : S_\epsilon^{m-1} \setminus K_\epsilon \rightarrow S_\epsilon^{m-1} \setminus K_\epsilon$ is a C^∞ -diffeomorphism given by the flow of ω in Lemma 3.2, and if $e^{is} \in S^1$, $h_t(\pi^{-1}(e^{is})) = \pi^{-1}(e^{i(s+t)})$, this says that this flow is transverse to all fiber $F_t := \pi^{-1}(t)$. Furthermore, if we consider U_α a neighborhood of $e^{i\alpha}$ in S^1 , small enough, we have the following commutative diagram:

$$\begin{array}{ccc}
U_\alpha \times \pi^{-1}(\alpha) & \xrightarrow{h_t} & \pi^{-1}(U_\alpha) \\
\pi_1 \downarrow & \swarrow \pi & \\
U_\alpha & &
\end{array}$$

where π_1 is the projection on first coordinate. □

5. COMPARING THIS RESULT WITH (c)–REGULARITY CONDITION

It is easy to see that the hypothesis of Theorem 1.6 implies (m)–condition for the pair (X, Y) , in a tubular neighborhood $T_Y^\epsilon = \{x \in \mathbb{R}^m \times \mathbb{R} : \rho(x, \theta) = \sum_{i=1}^m x_i^2 < \epsilon\}$, where $X = \Psi^{-1}(0) \setminus \{0\} \times \mathbb{R}$ and $Y = \{0\} \times \mathbb{R}$.

Corollary 5.1. *If the pair (X, Y) , defined as in Theorem 1.4, satisfy the Bekka’s (c)–condition on $\mathbb{R}^m \times \mathbb{R}$ with respect the control function $\rho(x, \theta) = \sum_{i=1}^n x_i^2$, then ψ satisfy the strong Milnor condition at origin.*

Proof. It turn out that Bekka’s (c)–regularity implies condition (m) and Whitney (a)–regularity condition for the pair of strata (X, Y) [BK]. □

The example below shows a real analytic maps germ which satisfies the hypothesis of Theorem 1.6 but the associated pair of strata (X, Y) does not satisfy Whitney (a)–regularity. More details can be found in [ACS]:

Example 5.2.

$$\begin{cases} P = x \\ Q = yx^2 + y^3 \end{cases}$$

Observe that $\Psi(x, y, \theta) = \cos(\theta)x - \sin(\theta)(yx^2 + y^3)$, so $\nabla \Psi_\theta(x, y) = (\cos(\theta) - 2xy \sin(\theta), -(x^2 + 3y^2) \sin(\theta))$. Let’s verify that in some punctured neighborhood of the origin the vectors $\nabla \Psi_\theta(x, y)$ and the position vector (x, y) are not parallel, for all $\theta \in \mathbb{R}$. It will be enough to solve the following system:

$$\begin{cases} \langle \nabla_{(x,y)} \Psi_\theta(x, y), (-y, x) \rangle = 0 \\ \Psi_\theta(x, y) = 0 \end{cases}$$

Or,

$$\begin{cases} y \cos(\theta) + (xy^2 + x^3) \sin(\theta) = 0 \\ x \cos(\theta) - (yx^2 + y^3) \sin(\theta) = 0 \end{cases}$$

The matrix of the system is

$$\begin{pmatrix} y & xy^2 + x^3 \\ x & -(yx^2 + y^3) \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The vector $(\cos(\theta), \sin(\theta))$ is always never zero, so we have to check the rank of the matrix on the right, but this determinant is $-y(yx^2 + y^3) - x(xy^2 + x^3) = -(x^2 + y^2)^2$. It is equal zero, iff $x = y = 0$. So, this map germ satisfies the hypothesis of main theorem.

In order to see that the associated pair (X, Y) of the map germ above does not satisfy the Whitney (a)–regularity, consider the sequences x_i, y_i, θ_i , with $x_i \rightarrow 0, y_i = 0, \theta_i = \frac{\pi}{2}$. It is easy to see that Whitney (a)–regularity fails along this sequence.

It means that the hypothesis given in Theorem 1.6 is weaker than the (c)–regularity condition over the pair (X, Y) given by the authors in [RS].

Example 5.3.

$$\begin{cases} P = z(x^2 + y^2 + z^2) \\ Q = y - x^3 \end{cases}$$

It is easy to see that this map germ has an isolated critical point at origin and the link is given by $K_\epsilon = \{z = 0, y = x^3\} \cap S_\epsilon^2$, i.e, two points. The points where $\nabla_{(x,y,z)} \Psi_\theta(x, y, z)$ and the vector position (x, y, z) are parallels satisfies the following system, for some $\lambda \in \mathbb{R}^*$:

$$\begin{cases} 2xz \cos(\theta) + 3x^2 \sin(\theta) = \lambda x \\ 2yz \cos(\theta) - \sin(\theta) = \lambda y \\ (x^2 + y^2 + 3z^2) \cos(\theta) = \lambda z \\ \cos(\theta)z(x^2 + y^2 + z^2) = \sin(\theta)(y - x^3) \end{cases}$$

Making some calculations you will get only the trivial solution.

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